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Publication date:
1996

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Discussion paper

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Tilburg University

Center
for
Economic Research

No. 96110

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December 1996

ISSN 0924-7815

WEAKLY CYCLIC GRAPHS AND DELIVERY GAMES

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November 8, 1996

Abstract

This paper studies a class of delivery problems associated with the Chinese postman problem and a corresponding class of delivery games. A delivery problem in this class is determined by a connected undirected (directed, mixed) graph, a cost function defined on its edges (arcs) and a special chosen vertex in that graph which will be referred to as the post office. It is assumed that the edges (arcs) in the graph are owned by different individuals and the delivery game is concerned with the allocation of the traveling costs incurred by the server, who starts at the post office, and is expected to traverse all edges in the graph before returning to the post office. A graph G is called Chinese postman-submodular, or, for short, CP-submodular (CP-totally balanced, CP-balanced, respectively) if for each delivery problem in which G is the underlying graph the associated delivery game is submodular (totally balanced, balanced, respectively).

For undirected graphs we show that CP-submodular graphs as well CP-totally balanced graphs turn out to be weakly cyclic graphs and conversely. An undirected graph is CP-balanced if and only if this graph is a weakly Euler graph. For directed graphs, CP-submodular graphs can be characterized by directed weakly cyclic graphs. Further, it is proven that each directed connected graph is CP-balanced. For mixed graphs it is shown that a graph is CP-submodular if and only if it is a mixed weakly cyclic graph.

Finally, we note that undirected, directed and mixed weakly cyclic graphs can be recognized in linear time.

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1 Introduction

A class of delivery games was introduced by *Hamers et al. (1994)* to analyze a cost allocation problem which arises in some delivery problems on graphs. These delivery problems are associated with the Chinese postman problem (*Mei-Ko Kwan (1962)*, *Edmonds* and *Johnson (1973)*) and can be described as follows. A server is located at some fixed vertex of a graph G, to be referred to as the post office, and each edge of G belongs to a different player. The players need some service, e.g. mail delivery, and the nature of this service requires the server to travel from the post office, and visits all edges (players) before returning to the post office. The cost allocation problem associated with this delivery problem is concerned with a fair allocation of the cost of a cheapest Chinese postman tour in the graph. That is, the cost of a cheapest tour, which starts at the post office, visits each edge of G at least once and returns to the post office. Following what is by now an established line of research, *Hamers et al. (1994)* formulated this cost allocation problem as a cooperative game (N, c), referred to as a delivery game, where N is the set of players (edges) in the graph, and $c : 2^N \rightarrow I\!R$ is the characteristic function. For each $S \subseteq N$, $c(S)$ is the cost of a minimal (i.e. cheapest) S-tour, which starts at the post office, visits each edge in S at least once and returns to the post office. Solution concepts in cooperative game theory were then evaluated as possible cost allocation schemes for the above delivery problem.

One of the most prominent solution concepts in cooperative game theory is the core of a game. It consists of all vectors which distribute the cost of a cheapest N-tour among the players in such way that no subset of players can be better off by seceding from the rest of the players and act on their own behalf. That is, a vector x is in the core of a game (N, c) if $\sum_{j \in N} x_j = c(N)$ and $\sum_{j \in S} x_j \leq c(S)$, for all $S \subset N$. A cooperative game whose core is not empty is said to be balanced, and if the core of any subgame of it has a nonempty core, it is said to be totally balanced.

In general, a delivery game associated with an undirected graph could have an empty core. However, *Hamers et al. (1994)* has shown if a connected graph is a weakly Euler graph, then the associated delivery game is balanced. Here, a graph G is called a weakly Euler graph if after the removal of the bridges in G the component are all Euler graphs or singletons. Further, *Hamers (1995)* has shown that if a connected undirected graph is weakly cyclic, that is, every edge therein is contained in at most one cycle, then the associated delivery game is submodular. That is, the characteristic funtion c is submodular.

In this paper we study the class of delivery games derived from undirected, directed and mixed graphs. We define a graph to be Chinese Postman-submodular, Chinese Postman-totally balanced or Chinese Postman-balanced (or, for short, CP-submodular, CP-totally balanced and CP-balanced), if the corresponding delivery game is submodular, totally balanced, or balanced, respectively, for all edge costs and all locations of the post office. We prove that an undirected graph G is CP-submodular if and only if it is CP-totally balanced, which holds if and only if G is weakly cyclic. Further, a undirected graph G is CP-balanced if and only if G is a weakly Euler graph. In contrast with the undirected case, we prove that any connected directed graph is CP-balanced. Further, we prove that a delivery game induced by a directed graph is submodular if and only if the directed graph is weakly cyclic. In a directed weakly cyclic graph each arc is contained in exactly one circuit. For a connected mixed graph, G is CP-submodular if and only if G is a mixed weakly cyclic graph. That is, each arc or edge is contained in at most one mixed circuit. Finally, we observe that undirected, directed and mixed weakly cyclic graphs can be recognized in linear time.

Our ability to characterize submodular delivery games is significant because submodular games are known to have nice properties, in the sense that some solution concepts for these games coincide and others have intuitive description. For example, for submodular games the Shapley value is the barycentre of the core (*Shapley (1971)*), the bargaining set and the core coincide, the kernel coincide with the nucleolus (*Maschler et al. (1972)*) and the τ-value (*Tijs (1981)*) can be easily calculated. Some examples of submodular games which were studied in the literature include airport games (*Littlechild* and *Owen (1973)*), tree games (*Megiddo (1978)*, *Granot et al. (1996)*), sequencing games (*Curiel et al. (1989)*, *Hamers et al. (1995)*) and certain communication games (*Van de Nouweland* and *Borm (1991)*.

Finally, we note that results obtained in this paper are in similar vein to those derived by *Herer* and *Penn (1996)* and *D. Granot, F. Granot* and *W.R. Zhu (1996)*. Therein, delivery games associated with the traveling salesman problem are investigated, and directed and undirected graphs which give rise to submodular delivery games for any edge costs and any starting vertex are characterized.

The paper is organized as follows. Section 2 introduces the delivery problem and the associated delivery game. Section 3 investigates the delivery game when G is undirected, and Section 4 is devoted to delivery games defined on directed graphs.

2 Delivery problems and delivery games

We present in this section a class of delivery problems associated with the Chinese postman problem and a corresponding class of delivery games. However, before a formal description of the models is presented, we need to provide some background in cooperative game theory and recall some elementary graph theoretical definitions.

A *cooperative (cost) game* is a pair (N, c), where N is a finite set of players, c is a mapping, $c : 2^N \to I\!R$, with $c(\emptyset) = 0$, and 2^N is the collection of all subsets of N. A subset of N will be sometimes referred to as a *coalition*. A function $h : 2^N \to I\!R$ is said to be *subadditive* if $h(S) + h(T) \geq h(S \cup T)$ whenever $S \cap T = \emptyset$ and it is said to be *submodular* if

$$h(T \cup \{j\}) - h(T) \leq h(S \cup \{j\}) - h(S) \tag{1}$$

for all $j \in N$ with $S \subset T \subseteq N \backslash \{j\}$. Equivalently, h is submodular if

$$h(S \cup T) + h(S \cap T) \leq h(S) + h(T) \tag{2}$$

for all coalitions $S, T \in 2^N$. A game (N, c) is submodular or concave if and only if the map $c : 2^N \to I\!R$ is submodular.

An allocation $x = (x_i)_{i \in N} \in I\!R^N$ is a *core-element* if $\sum_{i \in N} x_i = c(N)$ and $\sum_{i \in S} x_i \leq c(S)$ for all $S \in 2^N$. The *core* of a game (N, c) consists of all core elements. A game is called *balanced* if its core is non-empty and it is *totally balanced* if for each $S \subset N$, (S, c_S) is balanced, where c_S is the restriction of c to the family of subsets of S. It follows from *Shapley (1971)* that concave games are totally balanced.

Let $G = (V(G), E(G))$ be an undirected (directed) graph where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges (arcs) of G, respectively. An edge, $\{u, v\}$, in an undirected graph joins vertices u and v therein. If (u, v) is an arc from u to v in a directed graph (digraph), we will refer to u and v as the tail and head of arc (u, v), respectively. A *(directed) walk* in $G = (V(G), E(G))$ is a finite sequence of vertices and edges (arcs) of the form $v_1, e_1, v_2, ..., e_k, v_{k+1}$ with $k \geq 0, v_1, ..., v_{k+1} \in V(G), e_1, ..., e_k \in E(G)$ such that $e_j = \{v_j, v_{j+1}\}$ $(e_j = (v_j, v_{j+1}))$ for all $j \in \{1, ..., k\}$. Such a walk is said to be *closed* if $v_1 = v_{k+1}$. A *(directed) path* in G is a (directed) walk in which all vertices (except, possibly v_1 and v_{k+1}) and edges (arcs) are distinct. A closed (directed) path, i.e., a path in which $v_1 = v_{k+1}$, containing at least one edge (arc) is called a *(directed) circuit*. An undirected (directed) graph G is *connected* if there is a (directed) path from any vertex to any other vertex in G. An edge $b \in E(G)$ is called a *bridge* in a connected graph $G = (V(G), E(G))$ if the graph $(V(G), E(G) - \{b\})$ is not connected. The set of bridges in G is denoted by $B(G)$.

Let $G = (V(G), E(G))$ be a connected undirected (directed) graph, and let $v_0 \in V$ be an arbitrary vertex in $V(G)$, which will sometimes be referred to as a post office of G. An S-tour associated with $S \subseteq E(G)$ is a closed walk that starts in the post office v_0, visits each edge (arc) in S at least once and returns to v_0. Formally, we have:

Definition 2.1 *Let $G = (V(G), E(G), v_0)$ be a connected undirected (directed) graph in which $v_0 \in V(G)$ is the post office. An S-tour in G is a closed (directed) walk $v_0, e_1, v_1, ..., v_{k-1}, e_k, v_0$ such that $S \subseteq \{e_j \mid j \in \{1, ..., k\}\}$.*

The set of S-tours associated with $S \subseteq E(G)$ is denoted by $D(S)$.

Let $t : E(G) \to [0, \infty)$ be a travel cost function associated with edges (arcs) of G. The travel cost of an S-tour $v_0, e_1, v_1, ..., v_{k-1}, e_k, v_0$ is naturally equal to $\sum_{j=1}^{k} t(e_j)$.

The class of delivery problems we analyse in this paper and the corresponding class of cost allocations problems arise naturally in G when it is assumed that edges (arcs) therein belong to different players. Explicitly, assume that each edge (arc) in G belongs to a different player and that a server, located at v_0, is providing some service to players in G. The nature of this service in the delivery problem, which can be thought of as mail delivery, requires that the server will travel along the edges (arcs) of G and return to v_0. The corresponding cost allocation problem is concerned with the allocation of the cost of providing the service to the players.

Formally, let $\Gamma = (N(G), (V(G), E(G), v_0), t, g)$ denote a *delivery problem*, where $N(G)$ is the set of players, $(V(G), E(G), v_0)$ is a connected undirected (directed) graph in which v_0 represents the post office, $t : E(G) \to [0, \infty)$ assigns travel costs to the edges (arcs) and $g : E(G) \to N(G)$ is a one-to-one correspondence between the edges (arcs) and the players.

Definition 2.2 *The delivery game $(N(G), c)$ corresponding to the delivery problem $\Gamma = (N(G), (V(G), E(G), v_0), t, g)$ is defined for all $S \subseteq N(G)$ by*

$$c(S) = \min_{v_0, e_1, \ldots e_k, v_0 \in D(S)} \sum_{j=1}^{k} t(e_j). \tag{3}$$

Clearly, c is subadditive. Moreover, since the travel cost function is non-negative, delivery games are also monotonic, i.e. $c(S) \leq c(T)$ for all $S \subset T \subseteq N(G)$.

A delivery game (N, c) associated with a delivery problem Γ is totally balanced if for each $S \subset N(G)$ the subgame (S, c_S) is balanced. A graph G is said to be Chinese Postman-submodular, Chinese Postman-totally balanced or Chinese postman-balanced, or, for short, CP-submodular, CP-totally balanced, or CP-balanced, if for each delivery problem Γ in which G is the underlying graph the associated delivery game is submodular, totally balanced or balanced, respectively. Hence, if G is CP-submodular, CP-totally balanced or CP-balanced then for any choice of the travel cost function on G and any choice of the post-office in G, the corresponding delivery game is submodular, totally balanced, or balanced, respectively.

3 Weakly cyclic graphs, submodular graphs and totally balanced graphs: the undirected case

We characterize in this section CP-submodular graphs and CP-totally balanced graphs, when the underlying graph G in the delivery problem is assumed to be undirected.

Explicitly, we prove that both CP-submodular graphs and CP-totally balanced graphs are weakly cyclic graphs, where an undirected graph is said to be *weakly cyclic* if it is connected and every edge therein is contained in at most one circuit.

The first lemma shows that a necessary condition for a graph G to be CP-totally balanced is that G must be weakly cyclic.

Lemma 3.1 *A CP-totally balanced graph is weakly cyclic.*

PROOF: Let $(N(G),(G,v_0),t,g)$ be a delivery problem and suppose G is not weakly cyclic. Then, G contains a connected subgraph G^* of the form shown in Figure 1.1.

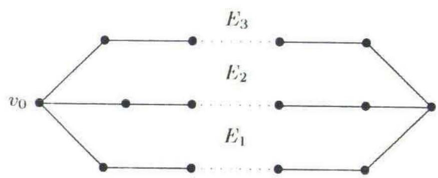

Figure 1.1.

Let S_1, S_2 and S_3 be the set of players associated with the edge sets E_1, E_2, and E_3, depicted in Figure 1.1, respectively. Let $N(G^*) = S_1 \cup S_2 \cup S_3$. Let v_0, as indicated in Figure 1.1, be the post office, and let t be a travel cost function satisfying $\sum_{e\in E_j} t(e) > 0$ for $j = 1, 2, 3$ and $t(e)$ is arbitrary large for $e \notin E_1 \cup E_2 \cup E_3$, and let $(N(G^*), c)$ be the subgame of $(N(G), c)$. We claim that with the above choice of v_0 and the cost function t, the core of $(N(G^*), c)$ is empty. Indeed, if the core is not empty, then there exists a vector $x, x \in I\!R^{N(G^*)}$, such that[3] $x(N(G^*)) = c(N(G^*))$ and

$$\begin{aligned} x(S_1 \cup S_2) &\leq t(E_1) + t(E_2) \\ x(S_1 \cup S_3) &\leq t(E_1) + t(E_3) \\ x(S_2 \cup S_3) &\leq t(E_2) + t(E_3). \end{aligned} \tag{4}$$

Summing the inequalities in (4) we obtain that

$$x(N(G^*)) \leq t(E_1) + t(E_2) + t(E_3) < c(N(G^*)),$$

where the last strict inequality follows since $t(E_i) > 0$ for $j = 1, 2, 3$. We have obtained a contradiction, since it was assumed that $x(N(G^*)) = c(N(G^*))$, and we conclude that $(N(G^*), c)$ is not balanced. Consequently, G is not CP-totally balanced. $\square$

Clearly, if G is CP-submodular, then G is also CP-totally balanced. Hence, from Lemma

[3] For a vector $y \in I\!R^N$ and $S \subseteq N$ we let $y(S) = \sum_{j\in S} y_j$.

3.1 it follows that a CP-submodular graph is weakly cyclic. The rest of this section is essentially devoted to prove that a weakly cyclic connected undirected graph is CP-submodular. First, we need to introduce some notation.

Let Γ be a delivery problem, let Q^m denote the edge set of a minimal Q-tour in Γ, and let Q^D consists of all distinct edges in Q^m. For simplicity we will denote by Q the player set corresponding to an edge set Q, instead of $g(Q)$. Let (N, c) be the delivery game corresponding to Γ. Then, the definition of Q^D implies

$$c(Q) = c(Q^D). \tag{5}$$

For any $Q, R \subseteq N(G)$ let $Q - R = \{j \in N(G) \mid j \in Q, j \notin R\}$. The following Lemma describes two simple CP-submodular graphs.

Lemma 3.2

(i) If a graph G consists of a single edge then G is CP-submodular.

(ii) If a graph G is a circuit then G is CP-submodular.

PROOF: The proof of (i) is trivial. A proof of (ii) is given in *Hamers (1995)*. For completeness, we provide below an alternative proof for (ii).

Let $\Gamma = (N(G), (G, v_0), t, g)$ be the delivery problem associated with G and let $(N(G), c)$ be the corresponding delivery game. We have to prove that $c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$ for all $S, T \subseteq N(G)$. We distinguish two cases:

Case 1: There exist minimal S-tour and T-tour in Γ such that $S^D \cap T^D = \emptyset$.
Since $S \subset S^D$ and $T \subset T^D$, it follows that $S \cap T = \emptyset$ and, consequently, $c(S \cap T) = 0$. Since c is subadditive, $c(S) + c(T) \geq c(S \cup T)$. Thus $c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$.

Case 2: For every minimal S-tour and T-tour in Γ, $S^D \cap T^D \neq \emptyset$.
Subcase 1: $S^D \subset T^D$.
Then, $c(S \cup T) = c(T)$, $c(S \cap T) \leq c(S)$, and $c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$.
Subcase 2: $S^D \not\subset T^D$ and $T^D \not\subset S^D$.
In this subcase either S^D and T^D can be partitioned, $S^D = S_1^D \cup S_2^D, T^D = T_1^D \cup T_2^D$ such that $S_1^D \subseteq T_1^D, T_2^D \subseteq S_2^D$, and subcase 1 can be applied to prove submodularity. Otherwise, we have in this subcase that $S^D \cup T^D = N(G)$ and $(S^D \cap T^D) \cup (S^D - T^D) \cup (T^D - S^D) = N(G)$. Moreover, it easy to verify that in this instance:

$$\begin{aligned} c(S) + c(T) &= \sum_{j \in S^D \cap T^D} 4t(j) + \sum_{j \in S^D - T^D} 2t(j) + \sum_{j \in T^D - S^D} 2t(j) \\ &\geq 2c(N(G)). \end{aligned}$$

Clearly, $c(S^D \cup T^D) = c(N(G))$ and $c(S \cap T) \leq c(N(G))$, which implies that $c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$. □

We need to introduce some new notation. Let $\Gamma = (N(G), (G, v_0), t, g)$ be a delivery problem and let (N, c) be the corresponding delivery game. For $v \in V(G)$, let $c(S; v)$ denote the cost of a minimal S-tour in G which also visits the vertex v. Let $\{v, w\}$ be an edge incident to vertex v in $G = (V(G), E(G))$. The *vertex-edge replacement* in G w.r.t. v results in a new graph, denoted G^v, derived from G by placing a new vertex v^* on the edge $\{v, w\}$. Thus $G^v = (V(G) \cup \{v^*\}, E^v(G))$ where $E^v(G) = E(G) \cup (\{v, v^*\}, \{v^*, w\}) \backslash (\{v, w\}))$. The delivery problem derived from $\Gamma = (N(G), (G, v_0), t, g)$, which corresponds to the vertex-edge replacement graph G^v, is $\Gamma^v = (N(G^v), (G^v, v_0), t^v, g^v)$, and will be referred to as the vertex-edge extension of Γ. Here, $t^v(e) = t(e)$ for all $e \in E(G) \backslash (\{v, w\})$, $t^v(\{v^*, w\}) = t(\{v, w\})$, $t^v(v, v^*) = 0$ and $g^v(e) = g(e)$ for all $e \in E(G) \backslash (\{v, w\})$,$g^v(\{v^*, w\}) = g(\{v, w\})$, $g^v(v, v^*) = n^v$. Thus, $N(G^v) = N(G) \cup \{n^v\}$.

Recall that for simplicity, we denote by Q the player set corresponding to an edge set Q. Therefore, consistent with our definition of g^v, for $S \subset E^v(G)$ such that $\{v^*, w\} \in S$ and $\{v, v^*\} \notin S$, $c(S)$ is the cost of a minimal $(S \backslash \{v^*, w\}) \cup (\{v, w\})$-tour in G. Let $N^v = N(G) \cup \{n^v\}$ and let (N^v, c^v) be the delivery game corresponding to Γ^v. Then it is easy to verify that for all $S \subseteq N(G)$

$$c^v(S \cup \{n^v\}) = c(S; v) \quad \text{and} \quad c^v(S) = c(S). \tag{6}$$

For a subset $Q \subseteq N^v$ we let $Q_v = Q \backslash \{n^v\}$. Further, recall that Q^m denotes the edge set of a minimal Q-tour in a graph and Q^D consists of all distinct edges in Q^m. From the definition of c^v and Q^D it follows that

$$c^v(Q) = c^v(Q^D). \tag{7}$$

Therefore, for any $Q \subseteq N^v$

$$c^v(Q) = c^v(Q^D) = c(Q_v^D), \tag{8}$$

where the first equality follows from (7) and the second equality follows from the second equality in (6) if $n^v \notin Q^D$, and, otherwise, if $Q^D = Q_v^D \cup \{n^v\}$, we can use the first equality in (6), since the vertex v is contained in Q_v^D.

By the definition of Γ and Γ^v it follows that for any $Q \subseteq N^v$ with $n^v \notin Q$

$$c(Q_v^D) = c(Q), \tag{9}$$

and for any $Q \subseteq N^v$ with $n^v \in Q$ we have that

$$c(Q_v^D) = c^v(Q^D) = c(Q_v^D; v) \geq c(Q_v; v), \tag{10}$$

where the first equality holds by (8), the second equality by (6) and the inequality holds since $Q_v \subseteq Q_v^D$.

Lemma 3.3 *If G is CP-submodular then G^v is also CP-submodular.*

PROOF: Let (N^v, c^v) be the delivery game corresponding to Γ^v and let (N, c) be the delivery game corresponding to Γ, with Γ^v being the vertex-edge extension of Γ. Let S

and T be arbitrary subsets in N^v. Then

$$\begin{aligned} c^v(S)+c^v(T) &= c(S_v^D)+c(T_v^D) \\ &\geq c(S_v^D\cup T_v^D)+c(S_v^D\cap T_v^D) \\ &= c((S^D\cup T^D)_v)+c((S^D\cap T^D)_v) \\ &\geq c((S\cup T)_v)+c((S\cap T)_v) \\ &= c((S\cup T)_v^D)+c((S\cap T)_v^D), \end{aligned}$$

where the first equality holds by (8), the first inequality holds by the CP-submodularity of G, the second inequality holds by monotonicity and the last equality holds by (5). Two cases will be considered.

Case 1: $n^v\in S\cap T$.

Clearly, in this case also $n^v\in S\cup T$ and it follows that

$$\begin{aligned} c((S\cup T)_v^D)+c((S\cap T)_v^D) &\geq c((S\cup T)_v;v)+c((S\cap T)_v;v) \\ &= c^v(S\cup T)+c^v(S\cap T), \end{aligned}$$

where the inequality holds by (10) and the equality holds by (6).

Case 2: $n^v\notin S\cap T$.

Subcase 1: $n^v\in S\cup T$.

Then

$$\begin{aligned} c((S\cup T)_v^D)+c((S\cap T)_v^D) &\geq c((S\cup T)_v;v)+c(S\cap T) \\ &= c^v(S\cup T)+c^v(S\cap T), \end{aligned}$$

where the inequality holds by (9) and (10) and the equality holds by (6).

Subcase 2: $n^v\notin S\cup T$.

Then

$$\begin{aligned} c((S\cup T)_v^D)+c((S\cap T)_v^D) &= c(S\cup T)+c(S\cap T) \\ &= c^v(S\cup T)+c^v(S\cap T), \end{aligned}$$

where the first equality holds by (9) and the second equality holds by (6).

Thus, we have proved that (N^v,c^v) is concave, which implies that the graph G^v is CP-submodular. □

Let $G_1=(V(G_1),E(G_1))$ and $G_2=(V(G_2),E(G_2))$ be two connected graphs with $V(G_1)\cap V(G_2)=\emptyset$. A 1-sum of G_1 and G_2 is obtained by coalescing one vertex in G_1 with another vertex in G_2. The newly formed vertex will be referred to as the 1-sum vertex.

Lemma 3.4 *Let the graph G_1+G_2 be a 1-sum of the connected graphs G_1 and G_2. If G_1 and G_2 are CP-submodular, then G_1+G_2 is also CP-submodular.*

PROOF: Let $\Gamma = (N(G_1+G_2), (G_1+G_2, v_0), t, g)$ be the delivery problem associated with G_1+G_2 and let $(N(G_1+G_2), c)$ be the corresponding delivery game. We need to show that $(N(G_1+G_2), c)$ is submodular for each location $v_0 \in V(G_1+G_2)$. For simplicity, we prove the result for $v_0 \in V(G_1)$.
For $i=1,2$, let $\Gamma_i = (N(G_i), (G_i, v_i), t_{E(G_i)}, g_{E(G_i)})$ and let $(N(G_i), c_{G_i})$ be the corresponding delivery game, with $v_1 = v_0$ and $v_2 = v$ is the 1-sum vertex in G_2. For $S \subseteq N(G_1+G_2)$, let $S_1 = S \cap N(G_1)$ and $S_2 = S \cap N(G_2)$. Then, for any $S \subseteq N(G_1+G_2)$,

$$c(S) = f^{G_1}(S_1) + c_{G_2}(S_2), \quad \text{where}$$

$$f^{G_1}(S_1) = \begin{cases} c_{G_1}(S_1) & \text{if} \quad S_2 = \emptyset \\ c_{G_1}(S_1; v) & \text{if} \quad S_2 \neq \emptyset. \end{cases} \tag{11}$$

Since $(N(G_2), c_{G_2})$ is submodular, the submodularity of $(N(G_1+G_2), c)$ would follow if we show that f^{G_1} is a submodular function.
Let S and T be arbitrary subsets in $N(G_1+G_2)$. We distinguish four cases:
Case 1: $S_2 = \emptyset$ and $T_2 = \emptyset$.
Then

$$\begin{aligned} f^{G_1}(S_1) + f^{G_1}(T_1) &= c_{G_1}(S_1) + c_{G_1}(T_1) \\ &\geq c_{G_1}(S_1 \cup T_1) + c_{G_1}(S_1 \cap T_1) \\ &= f^{G_1}(S \cup T) + f^{G_1}(S \cap T), \end{aligned}$$

where the two equalities hold by (11) and the inequality follows from the submodularity of $(N(G_1), c_{G_1})$.
Case 2: $S_2 \neq \emptyset$ and $T_2 \neq \emptyset$.
Then

$$\begin{aligned} f^{G_1}(S_1) + f^{G_1}(T_1) &= c_{G_1}(S_1; v) + c_{G_1}(T_1; v) \\ &= c^v_{G_1}(S_1 \cup \{v\}) + c^v_{G_1}(T_1 \cup \{v\}) \\ &\geq c^v_{G_1}(S_1 \cup T_1 \cup \{v\}) + c^v_{G_1}((S_1 \cap T_1) \cup \{v\}), \end{aligned}$$

where the first equality holds by (11), the second equality holds by (6) and the inequality follows from the submodularity of $(N(G_1) \cup \{v\}, c^v_{G_1})$ (cf. Lemma 3.3). Now, we have $c^v_{G_1}(S_1 \cup T_1 \cup \{v\}) = c_{G_1}(S_1 \cup T_1; v) = f^{G_1}(S_1 \cup T_1)$, since $S_2 \cup T_2 \neq \emptyset$. Further, $c^v_{G_1}((S_1 \cap T_1) \cup \{v\}) = c_{G_1}(S_1 \cap T_1; v)$, $c_{G_1}(S_1 \cap T_1; v) = f^{G_1}(S_1 \cap T_1)$ if $S_2 \cap T_2 \neq \emptyset$ and $c_{G_1}(S_1 \cap T_1; v) \geq f^{G_1}(S_1 \cap T_1)$ if $S_2 \cap T_2 = \emptyset$. Hence, $c_{G_1}(S_1 \cap T_1; v) \geq f^{G_1}(S_1 \cap T_1)$. The submodularity of f^{G_1} for this case follows.
Case 3: $S_2 \neq \emptyset$ and $T_2 = \emptyset$.
Then

$$\begin{aligned} f^{G_1}(S_1) + f^{G_1}(T_1) &= c_{G_1}(S_1; v) + c_{G_1}(T_1) \\ &= c^v_{G_1}(S_1 \cup \{v\}) + c^v_{G_1}(T_1) \\ &\geq c^v_{G_1}(S_1 \cup T_1 \cup \{v\}) + c^v_{G_1}(S_1 \cap T_1) \\ &= c_{G_1}(S_1 \cup T_1; v) + c_{G_1}(S_1 \cap T_1) \\ &= f^{G_1}(S_1 \cup T_1) + f^{G_1}(S_1 \cap T_1), \end{aligned}$$

where the first and fourth equalities hold by (11), the second and third equalities follow from (6), and the inequality follows from the submodularity of $(N(G_1) \cup \{v\}, c^v_{G_1})$ (cf. Lemma 3.3).

Case 4: $S_2 = \emptyset$ and $T_2 \neq \emptyset$.

The proof is identical to that of (iii).

Thus, we have proved above that f^{G_1} is submodular, which implies that $G_1 + G_2$ is CP-submodular. □

We are ready to present the following Theorem.

Theorem 3.1 *For an undirected graph G, the following statements are equivalent:*
(i) G is weakly cyclic.
(ii) G is CP-submodular.
(iii) G is CP-totally balanced.

PROOF: The case $(ii) \rightarrow (iii)$ holds since a submodular game is totally balanced and the case $(iii) \rightarrow (i)$ is already proved in Lemma 3.1. It remains to prove that $(i) \rightarrow (ii)$. Indeed, one can easily verify that a weakly cyclic graph can be obtained by 1-sums of circuits and single edges. By Lemma 3.2, the delivery games corresponding to circuits and single edges are submodular and by Lemma 3.4, a 1-sum of CP-submodular graphs is CP-submodular. □

Now, we will briefly discuss the recognition problem of a weakly cyclic graph. The connectedness of any graph can be checked in linear time. *Tarjan (1972)* showed that the biconnected components[4] of a graph can be found in linear time with respect to the number of vertices and edges. In a weakly cyclic graph, the biconnected components are the circuits. Since it can be checked in linear time whether a biconnected component is a circuit, we have proved the following proposition.

Proposition 3.1 *The computational complexity of determining whether a graph G is weakly cyclic is $\mathcal{O}(\mid E(G) \mid, \mid V(G) \mid)$.*

Hamers et al. (1994) discussed the CP-balancedness of the undirected case. They showed that if a connected undirected graph G is a weakly Euler graph then the graph is CP-balanced. Here, a graph G is called a weakly Euler graph if the components of the graph $(V(G), E(G) - B(G))$, the graph that arises from G by removing all bridges, are all Euler graphs or singletons. Recall that a graph is called an Euler graph if there exists a closed walk in that graph that visits each edge of this graph exactly once. We

[4]A biconnected component of a graph G is a maximal subgraph of G in which for each triple of distinct vertices v, w, z there exists a path between v and w not containing z.

refer to such a closed walk as an Euler tour. Before the next Theorem is formulated we need the following notation. For a path p we will denote by $V(p)$ and $E(p)$ the set of vertices and edges therein, respectively. The degree of a vertex $v \in V(G)$ in a graph $G = (V(G), E(G))$ is equal to the number of edges incident to that vertex v. The set $OV(G)$ denotes the set of vertices which have an odd degree in G.

Theorem 3.2 *A connected undirected graph G is weakly Euler if and only if G is CP-balanced.*

PROOF: If G is weakly cyclic, *Hamers et al. (1994)* have provided a vector that is in the core of the corresponding delivery game. So, here we have to prove the only if part. Suppose G is not a weakly Euler graph. Then there exists a component G^* in $G - B(G)$ that is not an Euler graph and not a singleton. This implies that $OV(G^*)$ is a non-empty set that contains an even number of vertices. Since G^* is connected, the vertices of $OV(G^*)$ can be covered by a forest G_c, which is a subgraph of G^*, in such a way that the graph that arises from G^* by multiplying the edges of G^* that correspond to G_c, is an Euler graph (cf. *Edmonds* and *Johnson (1973)*. Since G_c is a forest, there exists vertex-disjoint trees $T_1, ..., T_l$ such that the union of these trees is equal to G_c. Obviously, there exists a vertex in the tree T_1 such that the degree of v_1 is equal to one in T_1 and $v_1 \in OV(G^*)$. Since G^* contains no bridges, the degree of v_1 in G^* is at least three. Let $e_1, ..., e_k$ be all edges of $E(G^*)$ that are incident with v_1, and let $e_1 \in E(T_1)$. Since the degree of v_1 in T_1 is equal to one and the trees $T_1, ..., T_l$ are vertex-disjoint, we can conclude that $e_j \notin \cup_{i=1}^{l} E(T_i)$ for all $j \in \{2, ..., k\}$. Consider the cost function $t : E(G^*) \to [0, \infty)$ that is defined by $t(e_1) = 1 - \epsilon, 0 < \epsilon < 1, t(e_j) = 1$ for all $j \in \{2, ..., k\}$ and $t(e) = 0$ otherwise. Then the costs of a minimal $E(G^*)$-tour w.r.t. to v_1 is equal to $(k-1) + 2(1-\epsilon)$. This follows from the fact that G_c is the cheapest cover of G^* that yields an Euler graph. Any other cover that excludes e_1 has to use at least one of the edges $\{e_2, ..., e_k\}$, which implies that such a cover has at least costs 1, whereas the costs of G_c is equal to $1 - \epsilon$.

Let $v_2 \in OV(G^*), v_2 \neq v_1$ be incident to e_1. Since G^* is connected and contains no bridges, the graph $\underline{G}$ that arises from G^* by removing the edge e_1 is also connected. Then the forest $T_1', T_2, ..., T_l$, where $T_1' = (V(T_1) - \{v_1\}, E(T_1) - \{e_1\})$ is a cover of $OV(\underline{G})$ in $\underline{G}$ that yields an Euler graph. Since the costs of this cover is equal to zero we have that the costs of a minimal $E(\underline{G})$-tour in G^* w.r.t. v_1 is equal to $k - 1$.

Now, consider the graph $\hat{G}$, consisting of the edge set $E(G^*) - \cup_{j=1}^{k} E(T_j)$ and the vertices connected to the edges of this edge set. In the graph $\hat{G}$, which is not necessarily connected, the degree of each vertex is an even number. This implies that the components of $\hat{G}$ are singletons or Euler graphs. Since $e_j \notin \cup_{i=1}^{k} E(T_i)$ for all $j \in \{2, ..., k\}$ and k is an odd number greater of equal to three, we can conclude that v_1 is contained in a component of $\hat{G}$ that is an Euler graph. Let us describe the Euler tour of this component by t. Since G^* is connected and contains no bridges, there exists a path p from v_2 to some vertex h with $\{h\} = V(p) \cap V(t)$ and $e_1 \notin E(p)$. Note, that it is possible that $v_2 \in V(t)$. In this case we have that $v_2 = h$. Since t is an Euler tour there exists two subpaths t_1 and t_2 of t from h to v_1 such that $E(t_1) \cup E(t_2) = E(t)$ and $E(t_1) \cap E(t_2) = \emptyset$. For $j \in \{1, 2\}$, let G_{t_j} be the graph that consists of the edge e_1 and the paths p and t_j. Since G_{t_j} is a circuit, the minimal costs of a $E(G_{t_j}$-tour in G^* w.r.t. v_1 is equal to $k_j + (1 - \epsilon)$, where k_j is the degree of v_1 in the graph G_{t_j}. It is obvious that $k_1 + k_2 = k - 1$. Let $\overline{G_{t_2}})$ be the graph that consists of the edge e_1 and the path t_2. Then the minimal costs of a $E(\overline{G_{t_2}})$-tour is equal to $k_2 + (1-\epsilon)$. This holds since the costs of the minimal $E(G_{t_2})$-tour is equal to $k_2 + (1 - \epsilon)$ and the edges incident to v_1 are the same in as well $E(G_{t_2})$ as $E(\overline{G_{t_2}})$.

Now, we will partition the edges of $E(G^*) - \cup_{j=1}^{2} E(G_{t_j})$ in two sets $E(G_1^*)$ and $E(G_2^*)$. Let $e \in E(G^*) - \cup_{j=1}^{2} E(G_{t_j})$. If there exist a path q such that $e \in E(q)$, $E(q) \cap (E(t_1) \cup E(p)) \neq \emptyset$ and $E(q) \cap E(t_2) = \emptyset$, then $e \in E(G_1^*)$. Otherwise, we say $e \in E(G_2^*)$. For $j \in \{1, 2\}$, we have that the costs of all edges of $E(G_j^*)$ are equal to zero, and these edges can reach t_j by a path that contains only edges that have costs equal to zero. This implies that the costs of a minimal $(E(G_1^*) \cup E(G_{t_1}))$-tour in G^* w.r.t. v_1 is equal to $k_1 + (1 - \epsilon)$, and the costs of a minimal $(E(G_2^*) \cup E(\overline{G_{t_2}}))$-tour in G^* w.r.t. v_1 is equal to $k_2 + (1 - \epsilon)$.

Consider the following delivery problem $\Gamma = (N(G^*), (G^*, v_1), t, g)$ and let $(N(G^*), c)$ be the corresponding delivery game. Let the player sets corresponding to $E(G^*)$, $E(\underline{G})$, $E(G_1^*) \cup E(G_{t_1})$ and $E(G_2^*) \cup E(\overline{G_{t_2}})$ be $N(G^*)$, S_1, S_2 and S_3, respectively. From the values of the above minimal tours, we can conclude that

$$\begin{aligned} c(N(G^*) &= (k-1) + 2(1-\epsilon), \\ c(S_1) &= k - 1, \\ c(S_2) &= k_1 + (1-\epsilon) \text{ and} \\ c(S_3) &= k_2 + (1-\epsilon). \end{aligned}$$

By construction we have that $E(G^*) = E(G_1^*) \cup E(G_2^*) \cup E(t_1) \cup E(t_2) \cup E(p) \cup \{e_1\}$ and that the intersection of each pair of these edge sets is empty. This implies that $e^{N(G^*)} = \frac{1}{2}(e^{S_1} + e^{S_2} + e^{S_3})$, where $e_j^T = 1$ if $j \in T$ and $e_j^T = 0$ if $j \in N(G^*) - T$. We claim that the core of $(N(G^*), c)$ is empty. Indeed, if the core is not empty, then there exists a vector $x, x \in I\!R^{N(G^*)}$, such that $x(N(G^*)) = c(N(G^*))$ and

$$\begin{aligned} x(S_1) &\leq k - 1 \\ x(S_2) &\leq k_1 + (1-\epsilon) \\ x(S_3) &\leq k_2 + (1-\epsilon). \end{aligned} \tag{12}$$

Summing the inequalities in (12) we obtain that

$$2x(N(G^*)) \leq 2(k-1) + 2(1-\epsilon) < 2(k-1) + 4(1-\epsilon) = 2c(N(G^*)).$$

We have obtained a contradiction, since it was assumed that $x(N(G^*)) = c(N(G^*))$, and we conclude that $(N(G^*), c)$ is not balanced.

Finally, we will show that G is not CP-balanced. Consider the delivery problem $\Gamma' = (N(G), (G, v_1), t', g')$ where $t'(e) = t(e)$ if $e \in E(G^*)$ and $t'(e) = 0$, otherwise. Let $(N(G), c^*)$ be the delivery game corresponding to Γ'. We partition the edges of $E(G) - E(G^*)$ two sets $E(G_1)$ and $E(G_2)$. Let $e \in E(G) - E(G^*)$. If there exist a path q such that $e \in E(q)$, $E(q) \cap (E(G_{t_1}) \cup E(G_1^*)) \neq \emptyset$ and $E(q) \cap (E(\overline{G_{t_2}}) \cup E(G_2^*)) = \emptyset$, then $e \in E(G_1)$. Otherwise, we say $e \in E(G_2)$. Let the player set corresponding to $E(G_1)$ and $E(G_2)$ be $N(G_1)$ and $(N(G_2)$, respectively. Let $T_1 = S_1 \cup N(G_1) \cup N(G_2)$, $T_2 = S_2 \cup N(G_1)$ and $T_3 = S_3 \cup N(G_2)$. Now, it easy to verify that $c^*(N(G)) = c(N(G^*))$ and $c^*(T_j) = c(S_j)$ for $j \in \{1, 2, 3\}$. Now, we can prove the non-emptyness of the core of $(N(G)), c^*)$ in a similar way as we did for $(N(G^*), c)$. Hence, we can conclude that G is not CP-balanced.

□

4 Directed weakly cyclic graphs, submodular graphs and totally balanced graphs: the directed case

In this section it is assumed that the underlying graph of the delivery problem is a connected and directed. Observe that in a connected directed graph each arc is contained in at least one circuit. A connected digraph is said to be weakly cyclic if each arc is contained in precisely one directed circuit. In the following we will provide an alternative characterization of a directed weakly cyclic graph. For that purpose we need to introduce some new notation. Let G be a directed graph and let p' be a path from v_1 to v_2 in the underlying undirected graph associated with G. Let p be derived from p' by the introduction of the directions of edges in p' as they appear in G. If p is neither a directed path from v_1 to v_2, nor a directed path from v_2 to v_1, we will refer to p as *pseudo path*. A directed (pseudo) path p from a vertex v_1 to a vertex v_2 will be denoted by $p : v_1 \to v_2$ $(p : v_1 - v_2)$. Further, for a path p, we will denote by $V(p)$ and $E(p)$ the set of vertices and arcs theirin, respectively, and $I(p) = V(p) \backslash \{v_1, v_2\}$ will be refered to as the internal vertices of path p. Two path, p_1 and p_2, from v_1 to v_2 are called internally vertex-disjoint if $V(p_1) \cap V(p_2) = \{v_1, v_2\}$. Let $w_1, w_2 \in p_1$, where P_1 is either a directed path or a pseudo path. Then w_1 is closer to v_1 on p_1 than w_2, denoted by $w_1 \prec_{v_1, p_1} w_2$ if the (possibly pseudo) subpath $q : v_1 - w_1$ of p_1 does not contain w_2. The following lemma will be needed to provide a characterization of a weakly cyclic digraph.

Lemma 4.1 *Let G be a connected directed graph and let v_1 and v_2 be two different vertices in G. If there exist directed paths $p_1 : v_1 \to v_2$ and $p_2 : v_2 \to v_1$ and a pseudo path $p_3 : v_1 - v_2$ in G, then there exists an arc in G that is contained in at least two distinct directed circuits.*

PROOF: If p_3 is directed from v_1 to v_2 then each arc of p_2 is contained in the two circuits formed by p_1 and p_2 and by p_2 and p_3. Hence, we may assume that p_3 is not directed from v_1 to v_2, i.e. p_3 is a pseudo path with at least one arc directed towards v_1. Now, given p_3, we describe below a method to construct a directed path q from v_1 to v_2, with $q \neq p_1$. A generic step in this construction is as follows. For some $b_1^1 \in V(p_3)$ there exists a directed path $q_1 : v_1 \to b_1^1$, such that: (i) q_1 coincides with the subpath $\overline{p}_3 : (v_1 \to b_1^1)$ of p_3 and (ii) for the arc $(b_2^1, b_1^1) \in E(p_3)$ holds $b_1^1 \prec_{v_1,p_3} b_2^1$. Observe that b_2^1 cannot be reached directly from b_1^1 via p_3 since (b_2^1, b_1^1) is directed towards v_1. Also, b_1^1 could possibly coincide with v_1, in which case q_1 consists only of the vertex v_1. Now, since G is connected, there exists a directed path t_1 from b_1^1 to b_2^1. If $V(t_1) \cap (V(p_1) \cup V(p_2)) = \emptyset$, q_1 augmented with t_1 form a directed walk q_1' from v_1 to some vertex $w, w \in V(p_3)$, such that $b_2^1 \preceq_{v_1,p_3}$ and $I(q_1') \cap (V(p_1) \cup V(p_2)) = \emptyset$. This implies that there exists a directed path $\hat{q_1'} : v_1 \to w$ such that $I(\hat{q_1'}) \cap (V(p_1) \cup V(p_2)) = \emptyset$. We proceed now from vertex w along p_3 towards v_2 until we either reach v_2 or encounter an arc (b_2^2, b_1^2) such that $b_1^2 \prec_{v_1,p_3} b_2^2$. If we have reached v_2, then the structure consisting of $\hat{q_1'}, p_1$ and p_2 contains at least one arc which is contained in at least one directed circuit. Otherwise, we repeat the generic step to construct a directed path, t_2, from b_1^2 to b_2^2. If $V(t_2) \cap (V(p_1) \cup V(p_2)) = \emptyset$ we repeat the generic step. Eventually, if $V(t_j) \cap (V(p_1) \cup V(p_2)) = \emptyset$ for a sufficient number of path j, we will reach vertex v_2 and the conclusion that there exists at least one arc which is contained in at least two directed circuits. Thus, it remains to consider the case $V(t_1) \cap (V(p_1) \cup V(p_2)) \neq \emptyset$.

Let $h^* \in V(t_1) \cap (V(p_1) \cup V(p_2))$ be such that the directed subpath $t_1 : b_1^1 \to h^*$ contains no other vertex $h \in V(t_1) \cap (V(p_1) \cup V(p_2))$ and let $\hat{h} \in V(t_1) \cap (V(p_1) \cup V(p_2))$ be such that the directed subpath $t_1 : \hat{h} \to b_2^1$ contains no other vertex $h \in V(t_1) \cap (V(p_1) \cup V(p_2))$. We consider two cases.

Case 1: $(V(t_1) \cap (V(p_1) \cup V(p_2))) \cap \{v_1\} \neq \{v_1\}$.

If $h^* = v_1$, then from the assumption in Case 1 it follows that $h^* \neq \hat{h}$. Let p denote the directed path consisting of subpath $\overline{t_1}$ of t_1, $\overline{t_1} : \hat{h} \to b_2^1$, arc (b_2^1, b_1^1) and the directed subpath $\hat{t_1}$ of t_1, $\hat{t_1} : b_1^1 \to v_1$, define a directed path $p : \hat{h} \to v_1$ that is internally vertex-disjoint with p_1 and p_2. The structure consisting of p, p_1 and p_2 contains at least one arc that is contained in at least two directed circuits. Hence, we may assume that $h^* \neq v_1$. Let d_1 be the closest vertex to v_1 on q_1, such that $d_1 \in V(t_1)$. Formally, $d_1 \in V(q_1) \cap V(t_1)$ and if $\overline{q_1}$ denotes the subpath of q_1, $\overline{q_1}, : v_1 \to d_1$ then $I(\overline{q_1}) \cap (V(q_1) \cap V(t_1)) = \emptyset$. Since $h^* \neq v_1$, $\overline{q_1}$ contains at least one arc. Now, let p denote the directed path from v_1 to

h^* consisting of the directed subpath $\overline{q_1}$ and the directed subpath $\overline{t_1}$ of t_1, $\overline{t_1} : d_1 \to h^*$. By construction, $I(p) \cap (V(p_1) \cap V(p_2)) = \emptyset$. Thus, the structure consisting of p, p_1 and p_2 contains at least one arc that is contained in at least two directed circuits. We conclude therefore that if Case 1 occurs, there exists an arc that is contained in at least two directed circuits.

Case 2: $V(t_1) \cap (V(p_1) \cup V(p_2)) \cap \{v_1\} = \{v_1\}$.

In this case, the directed path which consists of the path q_1 augmented with t_1 forms a directed path from v_1 to b_1^2, which, possibly, has only the vertex v_1 in common with p_1 and p_2.

We then proceed from b_2^1 along p_3 towards v_2 until we encounter an arc (b_2^2, b_1^2) from b_2^2 to b_1^2 on p_3 such that $b_1^2 \prec_{v_1,p_3} b_2^2$ and repeat the generic step, where b_1^2 replaces b_1^1 and the directed path t_2 from b_1^2 to b_2^2 replaces the directed path t_1 from b_1^1 to b_2^1. Eventually, either for some $j \geq 2$ the directed path t_j^* satisfies Case 1 (with t_j replacing t_1), or, we have constructed a directed path $q : v_1 \to v_2$, such that q is internally vertex-disjoint with p_1 and p_2. The structure consisting of q, p_1 and p_2 contains at least one arc that is contained in at least two directed circuits, which completes the proof. □

The next Lemma provides alternative characterizations for a weakly cyclic graph.

Lemma 4.2 *Let G be a connected directed graph. Then, the following statements are equivalent:*

(i) G is weakly cyclic.

(ii) The underlying undirected graph G of G is a weakly cyclic graph that does not contain a bridge.

PROOF: $(i) \to (ii)$: Suppose G is weakly cyclic and assume first, on the contrary, that G is not weakly cyclic. Then there exist two vertices v_1 and v_2 that are connected by three internally vertex-disjoint paths. Hence, there exist in G three internally vertex-disjoint pseudo path, p_1, p_2 and p_3, between v_1 and v_2. Without loss of generality we may assume that p_1 is not a directed path from v_1 to v_2. Therefore, there exists an arc (b_1, b_2) from b_2 to b_1 such that $(b_2, b_1) \in E(p_1)$ and $b_1 \prec_{v_1,p_1} b_2$.The connectivity of G implies the existence of a directed path, t, from b_1 to b_2 and thus t augmented by (b_2, b_1) is a directed circuit, C, in G. Since $b_1, b_2 \in V(t) \cap V(p_1)$ and since p_1, p_2 and p_3 form three internally vertex disjoint (pseudo) paths between v_1 and v_2, we can conclude that there exists at least one arc (w_1, w_2') such that $w_1 \in V(t)$ and $(w_1, w_2') \in (E(p_1) \cup E(p_2) \cup E(p_3)) \backslash E(C)$, where $E(C)$ is the arc set of the circuit C. Therefore, there exists a pseudo path $q : w_1 - w_2$, which coincides with arc (w_1, w_2') if $w_2' \in V(t)$, such that $w_2 \in V(t)$ and $E(q) \subset (E(p_1) \cup E(p_2) \cup E(p_3)) \backslash E(C)$. The structure consisting of the directed circuit C containing vertices w_1 and w_2 and the pseudo path q between w_1 and w_2 implies, by Lemma 4.1, the exixtence of at least one arc in G which is contained in at least two directed circuits theirin. This contradicts our assumption that G is a weakly cyclic digraph.

To complete the proof we need to show that G does not contain a bridge. This follows from the fact that G is connected, and thus each arc therein is contained in at least one circuit. Hence, G cannot contain a bridge.

$(ii) \rightarrow (i)$: If $\overline{G}$ is weakly cyclic without bridges, then it is a 1-sum of undirected circuits. Thus, sine G is assumed to be connected, it must be a 1-sum of directed circuits, implying that every arc therein is contained in precisely one circuit. We conclude that G is a weakly cyclic digraph. □

From Lemma 4.2 we may conclude that a directed weakly cyclic graph can be obtained by 1-sums of directed circuits. The following Lemma shows that a CP-submodular graph is weakly cyclic.

Lemma 4.3 *A CP-submodular digraph is weakly cyclic.*

PROOF: Let $(N(G), (G, v_0), t, g)$ be a delivery problem and let (N, c) be the corresponding delivery game. Suppose G is not weakly cyclic. Then by definiton of a weakly cyclic digraph, there exists an arc (w_1, w_2) which is contained in two distinct directed circuits. This can be shown to imply the existence of three internally vertex-disjoint directed paths $p_1 : v_1 \rightarrow v_2$, $p_2 : v_2 \rightarrow v_1$ and $p_3 : v_2 \rightarrow v_1$. Let S_1, S_2 and S_3 be the set of players corresponding to the arcs contained in p_1, p_2, and p_3, respectively. Let v_1 be the post office, let $t(e) = 1$ for all arcs cointained in p_1, p_2 and p_3, and let $t(e) = \max\{| p_1 |, | p_2 |, | p_3 |\} + 1$ for all other arcs e, where $| p_j |, j = 1, 2, 3$ denotes the number of arcs in p_j. Then

$$\begin{aligned} c(S_1 \cup S_2 \cup S_3) + c(S_1) &= (2 \mid p_1 \mid + \mid p_2 \mid + \mid p_3 \mid) \\ &+ (\mid p_1 \mid + \min\{\mid p_2 \mid, \mid p_3 \mid\}) \\ &> (\mid p_1 \mid + \mid p_2 \mid) + (\mid p_1 \mid + \mid p_3 \mid) \\ &= c(S_1 \cup S_2) + c(S_1 \cup S_3), \end{aligned}$$

implying that c is not a submodular function. Hence, G is not CP-submodular. □

Let G be a weakly cyclic digraph and let v_0 be an arbitrary vertex therein. We can associate a directed tree $T(G, v_0)$ with (G, v_0) as follows. All arcs in the tree $T(G, v_0)$ are directed towards v_0, the root of the tree. A circuit in G, consisting of the arc set S, corresponds to an arc a_S in $T(G, v_0)$, and vertex v_S in $T(G, v_0)$ is the tail of arc a_S therein. Further, if two circuits, C_1 and C_2, consisting of arc sets S_1 and S_2 in G have a common vertex and the directed path from any node in C_1 to v_0 uses some arcs in C_2, then v_{S_2} is the head of arc a_{S_1} in $T(G, v_0)$. Let $\Gamma = (N(G), (G, v_0), t, g)$ be a delivery problem. Its corresponding directed tree problem is defined to be $\mathcal{T}, \mathcal{T} = \{N(G), T(G, v_0), t^*, g^*\}$, where $N(G)$ is the same player set as in Γ, $T(G, v_0)$ is the directed tree associated with (G, v_0) and t^* is the cost function in $T(G, v_0)$ satisfying $t^*(a_S) = \sum_{e \in S} t(e)$, for every directed circuit consisting of arcs S in G. The function g^* assigns the players corresponding to directed circuits in G to vertices in $T(G, v_0)$. Thus, if S is the set of arcs in a circuit of G, its corresponding vertex, v_S, in $T(G, v_0)$ contains the set of players S.

Let $(N(G), c)$ be the delivery game corresponding to $\Gamma = (N(G), (G, v_0), t, g)$ and let $(N(G), c^*)$ be the game corresponding to $\mathcal{T} = (N(G), T(G, v_0), t^*, g^*)$, where, for each $S \subset N(G)$, $c^*(S)$ is the total cost of all arcs in the minimal subtree of $T(G, v_0)$ that is rooted at v_0 and contains all vertices which contain players in S. By construction of the tree graph $T(G, v_0)$, there is a one-to-one correspondence between arcs in the tree and circuits in G. From this observation and the location of the players at vertices in the tree it follows that

$$c(S) = c^*(S) \text{ for all } S \subseteq N(G). \tag{13}$$

Display (13) implies that delivery games which arise from a weakly cyclic digraphs are contained in the class of tree games, introduced by *Megiddo (1978)*. *Granot, Maschler, Owen* and *Zhu (1996)* observed that tree games are submodular, which, in combination with Lemma 4.3, results in the following Theorem.

Theorem 4.1 *A connected digraph G is weakly cyclic if and only if G is CP-submodular.*

Meggido (1978) proved that for tree games Shapley value can be computed in $\mathcal{O}(n)$ and the nucleolus can be computed in $\mathcal{O}(n^3)$, where n is the number of vertices in the tree. *Galil (1980)* improved Meggido's algorithm and demonstrated that the nucleolus of a tree game can be computed in $\mathcal{O}(nlogn)$. *Granot, Maschler, Owen* and *Zhu (1996)* and *Potters, Maschler* and *Reijnierse (1996)* have developed other algorithms for computing the nucleolus of a tree games. Obviously, all these algorithms can be used to compute the nucleolus of delivery games that arise from CP-submodular digraphs.

Finally, we remark that one can easily construct examples of directed graphs for which the corresponding delivery games are totally balanced but not submodular. That is, in contrast with the undirected case, the class of CP-totally balanced directed graphs properly contains the class of CP-submodular directed graphs.

Moreover, the following Theorem demonstrates that, by contrast with the undirected case, a connected digraph is always CP-balanced.

Theorem 4.2 *A connected directed graph is CP-balanced.*

PROOF: Let G be a connected digraph, with an associated delivery problem $\Gamma = (N(G), (G, v_0), t, g)$ and a corresponding delivery game $(N(G), c)$. We have to show that $(N(G), c)$ is balanced.

For $S \subseteq N(G)$, consider the following linear programming (LP) problem:

$$c^*(S) = \min \sum_{i,j \in N(G)} t_{ij} x_{ij}$$

$$\text{subject to} \qquad (14)$$

$$\sum_{j \in N(G)} x_{ji} - \sum_{j \in N(G)} x_{ij} = 0 \text{ for all } i \in N(G)$$

$$x_{ij} \geq 1 \text{ for all arcs } (v_i, v_j) \in E(S),$$

$$x_{ij} \geq 0 \text{ for all arcs } (v_i, v_j) \notin E(S),$$

where t_{ij} denotes the cost of arc (v_i, v_j), x_{ij} denotes the flow in arc (v_i, v_j), and $E(S)$ is the set of arcs belonging to the players in S. For $S = N(G)$ an optimal solution for (14) is a minimum cost circulation on G such that the flow in each arc is at least one. In fact, the optimal value of (14) for $S = N(G)$ is the cost of an optimal Chinese postman tour in G with cost function t (cf. *Orloff (1974)*). Therefore, we conclude that $c^*(N(G)) = c(N(G))$. For $S \neq N(G)$ an optimal solution to (14) will consist of minimum cost circulations on G which may be disconnected. In fact, $c^*(S)$ is equal to the total cost of minimum cost (sub)tours that visit each arc of S at least once. In a minimal delivery tour of coalition S, each arc of S is also visited at least once. However, this tour has to be connected and must contain v_0. We conclude therefore that $c^*(S) \leq c(S)$ for all $S \subset N(G)$.

For a set of players (arcs) $S \subseteq N(G)$, let b^S denote the right hand side vector in (14). Then, one can easily verify that $b^S = \sum_{(i,j) \in S} b^{(i,j)}$, where $b^{(i,j)} = 1$ if $g((v_i, v_j)) \in S$ and $b^{(i,j)} = 0$ otherwise. Thus, (14) presents a linear production game formulation of $(N(G), c^*)$, and by *Owen (1975)* it follows that $(N(G), c^*)$ is totally balanced. Since $c^*(N(G)) = c(N(G))$ and $c^*(S) \leq c(S)$ for each $S \subset N(G)$, it follows that $(N(G), c)$ is balanced. $\square$

We note that it follows from Owen that if u_{ij} is an optimal dual variable associated with the lower bound constraint in the LP problem (14) associated with $S = N(G)$, then $u = ((u_{ij}) : (v_i, v_j) \in E(G))$ is in the core of the delivery game $(N(G), c)$. Therefore, it follows from *Tardos (1986)* that a core point in a delivery game associated with an arbitrary digraph can be found in strongly polynomial time.

Finally, we note that the recognition problem of a directed weakly cyclic graph G can be solved by considering the undirected underlying graph associated with G. Then essentially the same procedure for the recognition problem in the undirected case can be applied to the directed case. The only difference lies in the last step where one has to verify if each biconnected component is a directed circuit. However, this step can also be done in linear time. Hence, we conclude that the recognition of a directed weakly cyclic graph can be done in linear time. We conclude this section by considering briefly the case where the underlying graph $G = (V(G), E(G))$ is mixed. That is, an element in $E(G)$, which will be referred to as a connection, is either an arc or an edge. p is said to be a mixed path from v_1 to v_2 in G if the underlying undirected graph associated with p is a path between v_1 and v_2, and all arcs in p are directed from v_1 to v_2. A mixed circuit in G is defined similarly.

A connected mixed graph G is said to be weakly cyclic if each connection theirin is contained in at most one mixed circuit. Using a proof similar to the proof of Lemma 4.2, one can show that a connected mixed graph G is weakly cyclic if and only if the underlying undirected graph $\overline{G}$ of G is a weakly cyclic graph. Moreover, using similar techniques, one can proof the following result.

Theorem 4.3 *A connected mixed graph G is weakly cyclic if and only if G is a CP-submodular graph.*

As in the undirected case and directed case, mixed weakly cyclic graphs can be recognized in linear time. Finally, let us briefly consider the class of CP-totally balanced graphs in the mixed case. Clearly, by definition, a CP-submodular graph is CP-totally balanced. Our conjecture regarding the characterization of CP-totally balanced mixed graphs is as follows:

Conjecture 4.1 *Let G be a connected mixed graph. If G does not contain any of the three graphs in Figure 4.1 as an edge induced subgraph, then G is CP-totally balanced.*

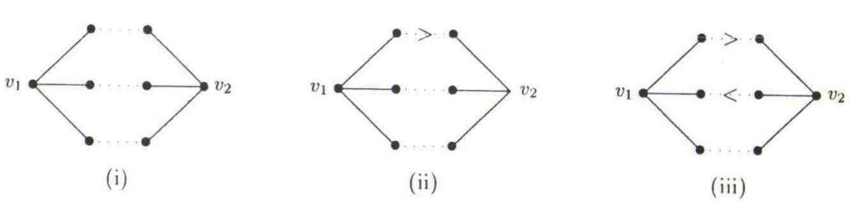

Figure 4.1

In all three cases in Figure 4.1, there are three internally vertex disjoint paths between v_1 and v_2. In case (ii), one of these three paths is a mixed path from v_1 to v_2, while in case (iii) one of the paths is a mixed path from v_1 to v_2 and another is the mixed path from v_2 to v_1.

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